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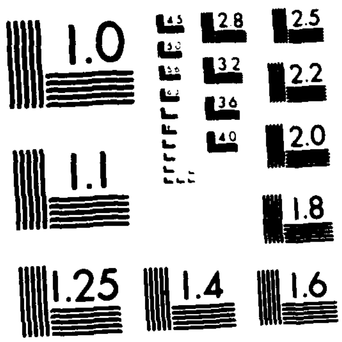

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MRC Technical Summary Report # 2566

HOMOGENIZATION FOR A VOLTERRA EQUATION

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September 1983

(Received March 2, 1983)

NOV 9

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ABSTRACT

A model is given for the non-linear heat equation in a heterogeneous medium with memory. Its homogenization is carried out in two particular cases (including the linear one).

AMS (MOS) Subject Classifications: 45D05, 73K20, 80A20, 45G10, 47H05.

Key Word: Volterra equation, homogenization, heat flow, heterogeneous material with memory

Work Unit Number 1 - Applied Analysis

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Nonlinear heat flow in a heterogeneous material is considered. In this model, the internal energy and heat flux depend upon the history of the temperature and the gradient of the temperature respectively.

The heat conservation law leads to a Nonlinear Volterra integro-differential equation with appropriate boundary conditions. This problem is solved under physically reasonable assumptions and its homogenization is investigated: introducing a small parameter ε measuring the "tightness" of the heterogeneity of the medium (typically we assume ε-periodicity for the physical parameters), the stability of the model is studied (as ε goes to zero) and the homogenized (ideal) limit medium is characterized in some cases, including the linear one.

The responsibility for the wording and views expressed in the descriptive summary lies with MRC, and not with the authors of this report.

HOMOGENIZATION FOR A VOLTERRA EQUATION

Hedy Attouch* and Alain Damlamian**

I. Introduction

In a heterogeneous medium with memory, a model for the heat equation (see Nohel [1]) is

$$\frac{\partial \xi}{\partial t} + \mathrm{div}_x\, Q = h_1 \quad , \tag{1.1}$$

where h_1 is a given diffused source term, ξ is the internal energy, Q is the heat flux. The latter are assumed to be functionals of the temperature distribution u with "memory":

$$\xi(t,x) = b_0(x)u(t,x) + \int_{-\infty}^{t} \beta(x,t-s)u(s,x)ds \tag{1.2}$$

$$\begin{aligned} Q(t,x) = -c_0(x)\sigma(x,\nabla u(t,x)) + \\ + \int_{-\infty}^{t} \gamma(x,t-s)\sigma(x,\nabla u(s,x))ds \quad . \end{aligned} \tag{1.3}$$

Equation (1.1) is considered on the product $\Omega \times (-\infty,T)$, where Ω is a bounded regular domain in $\mathbb{R}^3$ (or $\mathbb{R}^N$); the function $\sigma : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $(x,r) \to \sigma(x,r)$ represents a nonlinear flux law. Its dependence upon x specifies the heterogeneity of the medium. Similarly $b_0(x)$, $\beta(x,t)$, $c_0(x)$, $\gamma(x,t)$ characterize the spatial heterogeneity of the other thermodynamical parameters.

To equation (1.1) are added boundary and initial conditions which will be specified later.

The questions considered here are:

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- under what suitable set of hypotheses is equation (1.1) well-posed (existence and uniqueness);

- under what further conditions can one treat the corresponding homogenization problem; in other words, if all the parameters involved $(\sigma, b_0, \beta, c_0, \gamma)$ depend upon another variable ε measuring the "tightness" of the heterogeneity of the medium (typically $b_0^\varepsilon(x) = b_0(\frac{x}{\varepsilon})$ where $b_0(y)$ is periodic), can one find a limit problem whose solution would be the limit of the solutions u^ε, and whose structure would be similar to (1.1), (1.2), (1.3)?

We will give positive answers to both questions in particular cases only: in one case the nice method of Crandall-Nohel [1] applies and homogenization follows (§3, the splitting case); in §4, we deal with the general situation for which existence and uniqueness is proved via local monotonicity and global estimates; in §5 we treat the homogenization for the linear case via the Laplace transform.

A first approach to this type of problem appeared in Raynal [1].

II. Reformulation of the problem as a Volterra equation.

Let $*$ denote the usual convolution with respect to t on $[0,+\infty[$.

As "initial" condition, we assume the history of the medium to be known for t negative. One can then rewrite (1.1), (1.2), (1.3) as:

$$\frac{\partial}{\partial t}[b_0 u + \beta * u] + \mathrm{div}_x(-c_0\sigma + \gamma * \sigma) = h \tag{2.1}$$

where σ stands for $\sigma(\cdot, \nabla u(\cdot,\cdot))$ and the right hand side h includes the history of the system up to time zero.

It is customary to define

$$c(x,t) = c_0(x) - \int_0^t \gamma(x,s)ds \tag{2.2}$$

and to assume that $c(x,t)$ and $b_0(x)$ are strictly-positive valued (a physical condition). With these notations (2.1) can be written as:

(2.3) $$b_0 u' - \mathrm{div}_x\{(c*\sigma)'\} = h - \beta_0 u - \beta' * u \quad ,$$

where "'" indicates a time derivative.

The initial condition becomes a Cauchy data at $t = 0$:

(2.4) $$u(0) = u_0 \quad \text{in} \quad L^2(\Omega);$$

the boundary condition is taken to be compatible with the operator $-\mathrm{div}_x(\sigma(\nabla u))$, for example

(2.5) $$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \quad .$$

Problem (2.3), (2.4), (2.5), upon integration with respect to time appears as a Volterra type integral equation (cf. 3.1).

We now make precise the type of function σ appearing here: let $j : (x,r) \in \Omega \times \mathbb{R}^N \mapsto j(x,r) \in \mathbb{R}^+$, be of Caratheodory type, convex and equicoercive in r. Assume further that $j(x,0) \equiv 0$; let σ be its sub-differential with respect to r and put:

(2.6) $$\phi(u) = \begin{cases} \int_\Omega j(x,\nabla u(x))dx & \text{for} \quad u \in W_0^{1,1}(\Omega) \\ +\infty \quad \text{otherwise} & ; \end{cases}$$

then it is shown in Attouch-Damlamian [1] that ϕ is lower semi-continuous (l.s.c.) convex on $L^2(\Omega)$; moreover, denoting by $\partial\phi$ the subdifferential of ϕ on $L^2(\Omega)$ one has:

(2.7) $$v \in \partial\phi(u) \Longrightarrow \begin{cases} u \in W_0^{1,1}(\Omega) \cap L^2(\Omega) \\ v = -\mathrm{div}\, h \quad \text{for some} \quad h \quad \text{such that} \\ h(x) \in \sigma(x,\nabla u(x)) \quad \text{a.e.} \quad . \end{cases}$$

This is the sense in which σ is used in (2.3) and it also gives a meaning to the Dirichlet condition (2.5) (when j is even with respect to r, (2.7) is actually an equivalence).

III. The Splitting Case

By this, we mean that $\frac{c}{c_0}$ is independent of x. Equivalently, by an obvious change of notation (for σ), c and γ can be taken independent of x (c_0 is taken to be 1).

A) Existence and Uniqueness

Integration of (2.3) with respect to time from 0 to t yields:

$$b_0 u - \mathrm{div}_x(c*\sigma) = b_0 u_0 - \beta*u + H \tag{3.1}$$

where H is the integral of h.

(3.2) Proposition: Assume that b_0 and b_0^{-1} are in $L^\infty(\Omega)^+$, that β is in $L^\infty(\Omega; BV(0,T))$, and that γ is in $BV(0,T)$. Then equation (3.1) has a unique solution u in $C([0,T]; L^2(\Omega)) \cap L^1(0,T,W_0^{1,1}(\Omega))$. Furthermore, $\frac{du}{dt}$ is in $L^2(\Omega \times (0,T))$.

Proof: We follow here the ideas of Crandall - Nohel [1]. Let e be the resolvant kernel of γ, i.e. the solution of

$$e - \gamma - \gamma*e = 0 \quad . \tag{3.3}$$

Making use of standard results for convolution equations, one obtains that e belongs to $BV(0,T)$ as soon as γ does so. Using (3.3), (2.3) becomes

$$\begin{cases} b_0 \frac{du}{dt} - \mathrm{div}_x\, \sigma(x,\nabla u) = G(u) \\ u(0) = u_0 \ , \quad \text{where} \end{cases} \tag{3.4}$$

$$\begin{aligned} G(u) = {} & h + h*e - (\beta_0 + b_0 e_0)u + b_0 u_0 e - \\ & - u * (\beta_0 e + b_0 e' + \beta' + e*\beta';) \quad . \end{aligned} \tag{3.5}$$

Above $\beta_0 = \beta(0)$, $e_0 = e(0)$ $(= \gamma(0))$ and β', e'; are the measure derivatives of β and e.

It is easy to check that G is Lipschitz continuous from $L^1(0,t;L^2(\Omega))$ into itself for each $t > 0$. In order to apply a fixed

point theory (as in Crandall - Nohel [1]) we first consider the following problem for w in $C([0,T]; L^2(\Omega))$:

$$
(3.6) \qquad \begin{cases} b_0 \dfrac{du}{dt} - \mathrm{div}_x\, \sigma(x,\nabla u) = w \\ u(0) = u_0 \ . \end{cases}
$$

Now, the operator $u \to -\frac{1}{b_0}\,\mathrm{div}_x\, \sigma(x,\nabla u)$ is the subdifferential of ϕ (see (2.6)) on $L^2(\Omega)$ provided the norm on $L^2(\Omega)$ is chosen with the weight function $b_0(x)$ (since b_0 and b_0^{-1} are in $L^\infty(\Omega)^+$, this is an equivalent Hilbert norm on $L^2(\Omega)$). Therefore, (3.6) can be solved with classical estimates which allow to apply the Lipschitz fixed point theorem to solve (3.4), (3.5). ■

B] Homogenization

We now assume that b_0^ε, c_0^ε, γ^ε, β^ε and σ^ε depend upon an extra parameter ε measuring the size of the heterogeneity of the medium. A typical example is the periodic case where $b_0^\varepsilon(u) = \tilde{b}_0(\frac{x}{\varepsilon})$, $c_0^\varepsilon(x) = \tilde{c}_0(\frac{x}{\varepsilon})$, etc. ... where $\tilde{b}_0(y)$, $\tilde{c}_0(y)$,... are Y-periodic (Y is an N-dimensional parallelepipedon). We make the following hypotheses:

(3.7) $\qquad b_0^\varepsilon$ and $(b_0^\varepsilon)^{-1}$ are bounded in $L^\infty(\Omega)^+$,

β^ε is bounded in $L^\infty(\Omega; BV(0,T))$ and γ^ε is bounded in $BV(0,T)$.

Applying proposition (3.2) one gets:

(3.8) Proposition: Under hypothesis (3.7), there exists a unique solution u^ε for problem $(3.1)_\varepsilon$ and u^ε is bounded in $C([0,T]; L^2(\Omega))$ by a constant involving only $|h|_{L^2(\Omega)}$, $|\beta_0^\varepsilon|_{L^\infty(\Omega)}$, $|b_0^\varepsilon|_{L^\infty(\Omega)}$, $|(b_0^\varepsilon)^{-1}|_{L^\infty(\Omega)}$, $|\beta^\varepsilon|_{L^\infty(\Omega; BV(0,T))}$ and $|\gamma^\varepsilon|_{BV(0,T)}$. ■

In order to study the convergence of u^ε when ε goes to zero we make the following extra hypotheses (3.9) - (3.13):

(3.9) $j^\varepsilon(x,r)$ is coercive in r uniformly with respect to x and ε; more precisely:

$\exists K_1 > 0$, $\exists K_2 \in R$ and $p > \frac{2N}{N+2}$ such that

$$j^\varepsilon(x,r) \geq K_1 \, |r|^{\frac{2N}{N+2}} - K_2 \quad ,$$

(3.10) ϕ^ε (defined as in (2.6)) converge in the sense of Mosco[1] on $L^2(\Omega)$ to a limit denoted ϕ, which is then known to be of the same integral form, associated to a convex function j (cf. Attouch [1]). Finally $\phi(u_0)$ is assumed to be finite.

(3.11) β^ε converges in the weak-star topology of $L^\infty(\Omega; BV(0,T))$ to a limit β (consequently β_0^ε converges to β_0 in the weak star topology of $L^\infty(\Omega)$; in the periodic case, β is just the average of β^ε over a period).

(3.12) b_0^ε converges to some b_0 in the weak star-topology of $L^\infty(\Omega)$ and e^ε converges to some e in the weak-star topology of $BV(0,T)$. Then, e is the resolvant of some γ in $BV(0,T)$ (but γ has no relationship to the weak-star limit of γ^ε in $BV(0,T)$).

Consequently, the mapping G^ε (the analogue of G in (3.5)) is bounded so that $G^\varepsilon(u^\varepsilon)$ is bounded in $L^2(0,T; L^2(\Omega))$ and therefore, via the properties of the solutions of the problems $(3.6)_\varepsilon$, one can conclude (making also use of (3.10)) that

---

(1) For the definition and properties of this convergence, see Mosco [1] or Attouch [2].

(3.13) u^ε is bounded in $L^\infty(0,T; W_0^{1,p}(\Omega))$, $\frac{du^\varepsilon}{dt}$ is bounded in $L^2(0,T; L^2(\Omega))$, hence, as a consequence of Aubin's lemma (see Aubin [1]), u^ε is compact in $C([0,T]; L^2(\Omega))$.

We will show now that u^ε has only one possible limit value u when ε goes to zero, which implies that u^ε converges to u. Let therefore ε_n go to zero so that u^{ε_n} converges to some u in $C([0,T]; L^2(\Omega))$:

(3.14) <u>Proposition</u>: Under the above hypotheses ((3.13)), $G^{\varepsilon_n}(u^{\varepsilon_n})$ converges weakly in $L^2(\Omega \times (0,T))$ to $G(u)$ (as given in (3.5)), and $b_0^{\varepsilon_n} \frac{du^{\varepsilon_n}}{dt}$ converges weakly in $L^2(\Omega \times (0,T))$ to $b_0 \frac{du}{dt}$.

<u>Proof</u>: We write ε instead of ε_n for simplicity. $b_0^\varepsilon \frac{du^\varepsilon}{dt}$ is bounded in $L^2(Q)(Q = \Omega \times (0,T))$ and for $\varphi(x,t)$ in $\mathcal{D}(Q)$ one has:

$$\int_Q b_0^\varepsilon \frac{du^\varepsilon}{dt} \varphi = -\int_Q b_0^\varepsilon \frac{\partial \varphi}{\partial t} u^\varepsilon \longrightarrow -\int_Q b_0 \frac{\partial \varphi}{\partial t} u$$

which proves the second claim. For $G^\varepsilon(u^\varepsilon)$, each term can be treated independently; let us show convergence for the worst case: $w^\varepsilon = \beta^{\varepsilon'} * u^\varepsilon * e^\varepsilon$. First

$$w^\varepsilon(x,t) = \int_0^t e^\varepsilon(t-s) \int_0^s u^\varepsilon(x,s-\sigma)d\beta^\varepsilon(x,\sigma)ds$$

so that

$$|w^\varepsilon(x,\cdot)|_{L^\infty(0,T)} \leq |e^\varepsilon|_{L^1(0,T)} |u^\varepsilon(x,\cdot)|_{L^\infty(0,T)} |\beta^\varepsilon(x,\cdot)|_{BV(0,T)} ,$$

and

$$|w^\varepsilon|_{L^2(\Omega; L^\infty(0,T))} \leq |e^\varepsilon|_{L^1(0,T)} |u^\varepsilon|_{L^2(\Omega; L^\infty(0,T))} |\beta^\varepsilon|_{L^\infty(\Omega; BV(0,T))} .$$

Consequently, w^ε is bounded in $L^2(\Omega; L^\infty(0,T))$ because u^ε being bounded in $L^2(\Omega; H^1(0,T))$ is bounded in $L^2(\Omega; C([0,T]))$. By a similar argument, one checks that for almost every x in Ω,

$u^\varepsilon(x,t) \to u(x,t)$ in $C([0,T])$ which will be enough to show the convergence of w^ε to the proper w in $\mathcal{D}'(Q)$ as follows: for φ in $\mathcal{D}(Q)$

$$\langle w^\varepsilon,\varphi\rangle = -\int_\Omega\iiint \varphi'(x,t+s+\sigma)e^\varepsilon(t)u^\varepsilon(x,s)\beta^\varepsilon(x,\sigma) \; dt\, ds\, d\sigma\, dx$$

$$= \iint dtds\; e^\varepsilon(t)\iint_\Omega d\sigma\, dx\; \varphi'(x,t+s+\sigma)u^\varepsilon(x,s)\beta^\varepsilon(x,\sigma) \quad .$$

For every (t,s) the last integral converges to

$$\int d\sigma \int_\Omega \varphi'(x,t+s+\sigma)u(x,s)\beta(x,\sigma)dx$$

because β^ε converges to β *-weakly in $L^\infty(Q)$. Furthermore, Lebe s's dominated convergence theorem applies to the (t,s) integral since integrand is actually bounded by a constant, namely

$$|\varphi'|_\infty \cdot \sup|u^\varepsilon|_{L^\infty(0,T;\, L^2(\Omega))} \cdot \sup|\beta^\varepsilon|_{L^\infty(Q)} \quad ,$$

this last factor being bounded above by $\sup|\beta^\varepsilon|_{L^\infty(\Omega,BV(0,T))}$ which is finite by hypothesis (3.7). ■

Making now use of the convergence in the sense of Mosco of ϕ^ε to ϕ which implies a demi-closedness property (cf. Attouch [2]) one passes to the limit in

$$-\text{div}\;\sigma^\varepsilon(x,\nabla u^\varepsilon) = G^\varepsilon(u^\varepsilon) - b_0^\varepsilon \frac{du^\varepsilon}{dt}$$

to conclude that u is a solution of

$$\text{(3.15)} \qquad \begin{cases} -\text{div}\;\sigma(x,\nabla u) = G(u) - b_0 \dfrac{du}{dt} \\ u(0) = u_0 \quad . \end{cases}$$

Because G is as in (3.5) one concludes to the uniqueness of the solution for (3.15), hence the conclusion:

(3.16) <u>Theorem</u>: Under the hypotheses (3.7), (3.9), (3.10), (3.11), (3.12) (and with the notations therein), the solution u^ε of problem $(3.4)^\varepsilon$, $(3.5)^\varepsilon$

converges uniformly in $C([0,T]; L^2(\Omega))$ to the solution u of the analoguous problem whose thermodynamical parameters are obtained as follows:

β and b_0 are the weak-star limits of β^ε and b_0^ε respectively;

σ is the elliptic homogenization of σ^ε (equivalent to the Mosco convergence of ϕ^ε to ϕ cf. (3.10));

γ is the resolvant of the weak-limit of the resolvant of γ^ε (these two operations do not commute!). ■

IV. Existence and uniqueness in the general case

We start with equation (2.3) again

(4.1) $$b_0 u' - \operatorname{div}((c*\sigma)') = g(u),\ u(0) = u_0 \quad ,$$

where

(4.2) $$g(u) = h - \beta_0 u - \beta' * u$$

is Lipschitz continuous from $L^1(0,t; L^2(\Omega))$ into itself for every positive t. Therefore, one can solve (4.1) as a Lipschitz perturbation problem (in a fashion similar to that in III A)) provided one can solve

(4.3) $$\begin{cases} b_0 \dfrac{du}{dt} - \operatorname{div}((c*\sigma)') = F \\ u(0) = u_0 \end{cases}$$

for given F, via a monotonicity argument.

Here, one should notice that the method of Crandall - Nohel does not apply because c depends upon x so that div and convolution with c do not commute.

We make the following assumptions where α and k are given positive numbers:

(4.4) i) $b_0, \dfrac{1}{b_0}, c_0, \dfrac{1}{c_0}, \beta_0, \dfrac{1}{\beta_0}$ are bounded by k in $L^\infty(\Omega)$ and $|u_0|_{L^2(\Omega)} < k$.

ii) $\beta' < 0$ and $c' = -\gamma < 0$ a.e. in Q; $\beta(T,x)$ and $c(T,x)$ are bounded below away from zero by α; $t \mapsto c(t,x)$ is continuous at $t = 0$ with values in $L^\infty(\Omega)$.

iii) β'' and $c''(= -\gamma')$ are nonnegative measures for a.e. x in Ω.

iv) (2.6) holds with the following inequalities

$\forall$ r,s in $\mathbb{R}^N$,

$\alpha|r-s|^2 < (\sigma(x,r) - \sigma(x,s), r-s)$

and $|\sigma(x,r) - \sigma(x,s)| < k|r-s|$.

We start by choosing t small enough as follows:

(4.5) <u>Proposition</u>: Let $Au = -\mathrm{div}((c*\sigma)') = -\mathrm{div}(c_0\sigma) - \mathrm{div}(c'*\sigma)$. For T small enough, A is maximal monotone from $L^2(0,T;\ H_0^1(\Omega)) = V$ into $L^2(0,T);\ H^{-1}) = V'$.

<u>Proof</u>: We estimate

$$
\begin{aligned}
&(Au-Av,u-v)_{V',V} > \\
&\quad > \int_Q c_0(\sigma(x,\nabla u(x) - \sigma(x,\nabla v(x)),\ \nabla u(x) - \nabla v(x))dxdt \\
&\qquad + \int_Q (c'*(\sigma(x,\nabla u(x)) - \sigma(x,\nabla v(x)),\ \nabla u(x) - \nabla v(x))dxdt \\
&\quad > \alpha \int_Q c_0|\nabla u-\nabla v|^2 dxdt \\
&\qquad - \int_Q |c'|*|\sigma(\nabla u) - \sigma(\nabla v)|\ |\nabla u-\nabla v| dxdt \\
&\quad > \alpha \int_Q c_0|\nabla u-\nabla v|^2 dxdt \\
&\qquad - \int_\Omega dx|c'(x)|_{L^1(0,T)} |\sigma(\nabla u)-\sigma(\nabla v)|_{L^2(0,T)} |\nabla u-\nabla v|_{L^2(0,T)} \\
&\quad > \alpha \int_Q c_0|\nabla u-\nabla v|^2 dxdt - k|c'|_{L^\infty(\Omega;\ L^1(0,T))} \int_Q |\nabla u-\nabla v|^2 dxdt .
\end{aligned}
$$

Note now that $|c'(x)|_{L^1(0,T)} = c_0(x) - c(t,x)$ so by (4.4) ii)

$|c'|_{L^\infty(\Omega;\ L^1(0,T))}$ is arbitrarily small for T arbitrarily close to zero.

For such a T, $(Au-Av,u-v) > \theta|\nabla u-\nabla v|^2_{L^2(Q)}$ for some positive number θ.

Hence A is monotone. Being also everywhere defined and continuous on V, it is maximal. ■

Now, for such a small T, $b_0 \frac{d}{dt} + A$ is one to one and onto from V to V' (because of a standard non-linear argument of coerciveness, cf. Brezis [1]). This proves local existence and uniqueness of the solution for problem (4.3), hence for (4.1), (4.2). In order to prove global existence we now get two a priori estimates.

(4.6) <u>Proposition</u>: Under the above hypotheses, there is a constant C_1 (depending upon α and k) such that if u is a solution of (4.1), (4.2), the following holds:

$$|u|_{L^\infty(0,T;L^2(\Omega))} < C_1 \quad \text{and} \quad |j(\nabla u)|_{L^1(Q)},\ |j^*(\sigma(\nabla u))|_{L^1(Q)} < C_1$$

(consequently $|\nabla u|^2_{L^2(Q)} < kC_1$, $|\sigma(\nabla u)|^2_{L^2(Q)} < \frac{C_1}{\alpha}$).

<u>Proof</u>: Multiply (4.1), (4.2) by $u(t)$ and integrate by parts to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|b_0^{1/2}u(t)|^2_{L^2(\Omega)} + |\beta_0^{1/2}u(t)|^2_{L^2(\Omega)} \\ (4.7) \qquad &+ \int_\Omega(\beta'*u)(t)u(t)dx + \int_\Omega(c_0\sigma+c'*\sigma)(t)\cdot\nabla u(t)dx = \\ &= \int_\Omega h(t)u(t)dx \ . \end{aligned}$$

Integrating on $(0,t)$ yields

$$\begin{aligned} &\frac{1}{2}|b_0^{1/2}u(t)|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \beta_0 u^2 dsdx + \int_0^t\int_\Omega c_0\sigma(\nabla u(s))\nabla u(s)dsdx \\ (4.8) \qquad &= \int_0^t\int_\Omega h\cdot u\, dsdx + \int_0^t\int_\Omega(-\beta'*u)(s)u(s)dsdx + \\ &+ \int_0^t\int_\Omega(-c'*\sigma)(s)\nabla u(s)dsdx + \frac{1}{2}|b_0^{1/2}u_0|^2_{L^2(\Omega)} \ . \end{aligned}$$

By (4.4) ii), one has

$$
(4.9)\begin{cases}
\int_0^t(-\beta'*u)(s)u(s)ds \le \int_0^t \int_0^s -\beta'(s-\tau)(\frac{1}{2}|u(s)|^2 + \frac{1}{2}|u(\tau)|^2)dsd\tau \\[2ex]
\qquad \le \int_0^t \frac{1}{2}|u(s)|^2(\beta_0-\beta(s))ds + \frac{1}{2}\int_0^t ds \int_0^s -\beta'(s-\tau)|u(\tau)|^2 d\tau \\[2ex]
\qquad \le \frac{1}{2}\int_0^t(\beta_0-\beta(s))|u(s)|^2 ds + \frac{1}{2}(\int_0^t -\beta'(\tau)d\tau)(\int_0^t|u(\tau)|^2 d\tau) \\[2ex]
\qquad \le (\beta_0-\beta(t))\int_0^t u^2(s)ds \quad .
\end{cases}
$$

Hence

$$
(4.10) \qquad \int_0^t \int_\Omega \beta_0 u^2 + (\beta'*u)u\, dxds \ge \int_\Omega \beta(x,t)\int_0^t|u(x,s)|^2 dsdx \quad .
$$

A similar computation, making use of Young's inequality corresponding to j and j^*, gives:

$$
(4.11) \quad \begin{aligned}
&\int_0^t \int_\Omega(c_0\sigma(\nabla u) + c'*\sigma(\nabla u)) \cdot \nabla u\, dxds \ge \\
&\qquad \ge \int_\Omega c(x,t)\int_0^t j(x,\nabla u(x,s)) + j^*(x,\sigma(x,\nabla u(x,s))dsdx \quad .
\end{aligned}
$$

Now (4.8), (4.10) and (4.11) combined give

$$
(4.12)\quad \begin{cases}
\frac{1}{2}|b_0^{1/2}u(t)|^2_{L^2(\Omega)} + \int_\Omega \beta(t,x)\int_0^t u^2(s,x)dsdx + \\[2ex]
\qquad + \int_\Omega c(t,x)\int_0^t(j(s,\nabla u) + j^*(x,\sigma(\nabla u)))dsdx \le \\[2ex]
\qquad \le \int_0^t \int_\Omega h\, u\, dsdx + \frac{1}{2}|b_0^{1/2}u_0|^2_{L^2(\Omega)} \quad .
\end{cases}
$$

A standard application of Gronwall's inequality finally yields the desired result.

(4.13) <u>Proposition</u>: Assume the above hypotheses and that $\phi(u_0)$ is finite. Then, there is a constant C_2 (depending upon α, k and $\phi(u_0)$) such that whenever u is a solution of (4.1), (4.2) then:

$$
|\frac{du}{dt}|_{L^2(Q)} \le C_2 \quad , \quad |\phi(u)|_{L^\infty(0,T)} \le C_2 \quad .
$$

<u>Proof</u>: Multiply (4.1), (4.2) by $\frac{du}{dt}$ to get

$$\int_0^t \int_\Omega b_0 \left|\frac{du}{dt}\right|^2 dxds + \int_0^t \int_\Omega (\beta_0 u + \beta' * u) \frac{du}{dt}(s) dxds + \tag{4.14}$$

$$+ \int_0^t \int_\Omega (c_0\sigma + c' * \sigma) \frac{d}{dt}(\nabla u(s)) dxds = \int_0^t \int_\Omega h \frac{du}{dt}(s) dxds \quad .$$

In (4.14) we integrate by parts the third term as follows:

$$\int_0^t (\beta' * u)u' = u(t) \int_0^t u(s)\beta'(t-s)ds - \beta'(0) \int_0^t u^2(s)ds$$

$$- \int_0^t u(s) \int_0^s u(\sigma)\beta''(s-\sigma)d\sigma ds \quad .$$

Making use of $\beta'' > 0$, one can evaluate the last term in a way exactly similar to (4.9) above to get

$$\int_0^t (\beta' * u) \frac{du}{dt} > u(t) \int_0^t u(s)\beta'(t-s)ds - \beta'(0) \int_0^t u^2(s)ds \quad . \tag{4.15}$$

Again, similarly,

$$\int_0^t (c' * \sigma) \frac{d}{dt}(\nabla u)ds > \nabla u(t) \cdot (c' * \sigma)(t) - c'(0) \int_0^t \sigma(s)\nabla u(s)ds \quad . \tag{4.16}$$

Using (4.15), (4.16) and the following consequence of the definition of sub-differential $\sigma = \partial j$:

$$\sigma \cdot \frac{d}{dt}(\nabla u) = \frac{d}{dt} j(\nabla u) \quad , \tag{4.17}$$

(4.14) yields

$$\int_0^t \int_\Omega b_0 \left|\frac{du}{dt}\right|^2 dxds + \frac{1}{2} |\beta_0^{1/2} u(t)|^2_{L^2(\Omega)} +$$

$$+ \int c_0 j(\nabla u(t))dx < \int_0^t \int_\Omega h \frac{du}{dt} dxds +$$

$$+ \frac{1}{2} |\beta^{1/2} u_0|^2_{L^2(\Omega)} + \int_\Omega c_0 j(\nabla u_0)dx + \tag{4.18}$$

$$+ \int_\Omega u(x,t) \int_0^t u(x,s)(-\beta'(x,t-s))dxds$$

$$+ \int_\Omega \nabla u(x,t) \int_0^t \sigma(x,\nabla u(x,s))(-c'(x,t-s))dsdx \quad .$$

In the right hand side of (4.18) one can use the following bounds:

$$\text{i)} \quad \int_\Omega \int_0^t u(t)u(s)(-\beta'(t-s))dxds <$$

$$< |u|^2_{L^\infty(0,T;L^2(\Omega))} \cdot |\beta'|_{L^1(0,T;L^\infty(\Omega))} \quad . \tag{4.19}$$

ii) By Young's inequality

$$\int_\Omega \int_0^t \sigma(s)\nabla u(t) \cdot (-c'(t-s))dsdt$$
$$< \int_\Omega j(\nabla u(t))(c_0-c(t)) + |c'|_{L^\infty(Q)} |j^*(\sigma(\nabla u))|_{L^1(Q)} .$$

Now, confronting (4.18) and (4.19) with hypothesis (4.4) ii) and proposition (4.6) one can conclude. ■

(4.20) <u>Theorem</u>: Under hypotheses (4.4), problem (4.1), (4.2) has a unique solution on $[0,T]$.

<u>Proof</u>: From proposition (4.5), there is existence and uniqueness on some interval $[0,t]$ $t > 0$. But the very same proposition gives existence and uniqueness locally in time (starting from $\tau > 0$, the problem is changed only insofar as G in (4.2) is modified to incorporate the history up to τ; this in no way changes the conclusion because of the a priori estimates are global in time). Combining local existence and the a priori estimates (4.6) and (4.13) gives the result in a standard way. ■

V. <u>Homogenization (for the linear case)</u>

In this paragraph, b_0^ε, β^ε, c^ε and γ^ε will depend upon the parameter ε which will tend to zero; similarly, σ^ε will depend upon ε but will be assumed to be linear with respect to ∇u, hence the notation

$$\sigma^\varepsilon(x,r) = A^\varepsilon(x)r \quad , \quad \text{where} \quad A^\varepsilon(x)$$

is a measurable function from Ω to a fixed (independent of ε) set of symmetric uniformly positive definite matrices.

We shall assume hypotheses (4.4) to be satisfied uniformly with respect to ε, so that (4.6), (4.13) and (4.20) hold uniformly in ε.

Consequently, the solutions u^ε of equations $(4.1)_\varepsilon$, $(4.2)_\varepsilon$, belong to a compact set of $C([0,T]; L^2(\Omega))$ and a bounded set of

$$L^\infty(0,T;\ H_0^1(\Omega)) \cap W^{1,2}(0,T;\ L^2(\Omega)) \quad .$$

The question of homogenization for $(4.1)_\varepsilon$, $(4.2)_\varepsilon$ is: what can be said of the limit points of u^ε as ε goes to zero?

In order to simplify the notations we shall assume that ε belongs to a sequence (still denoted ε) such that the following holds:

(5.1) u^ε converges to some u in $C([0,T];\ L^2(\Omega))$, in the weak-star topology of $L^\infty(0,T;\ H_0^1(\Omega))$ and the weak topology of $W^{1,2}(0,T;\ L^2(\Omega))$;

(5.2) b_0^ε converges to b_0 in the weak-star topology of $L^\infty(\Omega)$; β^ε converges to β in

$$\sigma(W^{1,1}(0,T;\ L^\infty(\Omega)),\ W^{-1,\infty}(0,T;\ L^1(\Omega))) \quad .$$

Following integration in t, the Volterra equation can be written as

$$- \operatorname{div} \mathcal{W}^\varepsilon = F^\varepsilon \quad , \tag{5.3}$$

where

$$\mathcal{W}^\varepsilon(x,t) = \int_0^t A^\varepsilon(x)c^\varepsilon(x,t-s)\nabla u^\varepsilon(x,s)ds \tag{5.4}$$

and

$$\begin{aligned} F^\varepsilon(x,t) = -b_0^\varepsilon(x)u^\varepsilon(x,t) - \int_0^t \beta^\varepsilon(x,t-s)u^\varepsilon(x,s)ds \,+ \\ + \int_0^t h(x,s)ds + b_0^\varepsilon(x)u_0(x) \quad . \end{aligned} \tag{5.5}$$

Clearly F^ε converges to F in $C([0,T],\ H^{-1}(\Omega))$ and weakly in $W^{1,2}(0,T;\ L^2(\Omega))$, for

$$F(x,t) = b_0(x)(u_0(x)-u(x,t)) + \int_0^t h(x,s)ds - \int_0^t \beta(x,t-s)u(x,s)ds \quad . \tag{5.6}$$

On the other hand, $\mathcal{W}^\varepsilon$ is bounded in $W^{1,2}(0,T;\ L^2(\Omega))$. In order to characterize the possible weak limits $\mathcal{W}$ of $\mathcal{W}^\varepsilon$ (the uniqueness of which will follow, as usual from the unique solvability of the limit equation), we shall assume, after extracting another subsequence, still denoted ε, that $\mathcal{W}^\varepsilon$ converges weakly to some $\mathcal{W}$. So, going to the limit in (5.3) yields:

$$(5.7) \qquad - \operatorname{div} \mathcal{W} = \mathcal{F} \quad .$$

The main task is to find the relationship between $\mathcal{W}$ and u, which (5.4) should yield. Here, because the problem is linear, we use the Laplace transform, but to do so we extend the problem to $[0,+\infty)$ in time as follows: for $t > T$, extend β^ε by $\beta^\varepsilon(x,t) \equiv \beta^\varepsilon(x,T)$, h and γ^ε by zero (so $c^\varepsilon(x,t) \equiv c^\varepsilon(x,T)$), and by theorem (4.20) which applies to any interval $[0,T_1]$, u^ε exists for all $t > 0$, but for $t > T$, the problems become simpler, as seen from (2.1):

$$b_0^\varepsilon \frac{du^\varepsilon}{dt} + \beta^\varepsilon(T)u^\varepsilon - \operatorname{div}_x A(c_0 \nabla u^\varepsilon - \gamma * \nabla u^\varepsilon) = 0 \quad .$$

A detailed analysis of estimates (4.6), (4.13) shows that in this particular case, $|u^\varepsilon(t)|_{H^1(\Omega)}$ grows at most exponentially in t with a rate uniform in ε. Therefore, all the Laplace transforms considered here will be convergent at least in some complex right half-plane $\operatorname{Re}\lambda > \lambda_0$.

We will denote by

$$\hat{v}(\lambda) = \int_0^{+\infty} e^{-\lambda t} v(t)dt \quad \text{for} \quad \operatorname{Re}\lambda > \lambda_0$$

and (5.3), (5.4) yield $(5.8)_\varepsilon$ below since the gradient operator in x commutes with the Laplace transform.

$$(5.8)_\varepsilon \quad \begin{cases} \hat{\mathcal{W}}^\varepsilon(x,\lambda) = A^\varepsilon(x)\ \hat{c}_\varepsilon(x,\lambda)\ \nabla(\hat{u}_\varepsilon(x,\lambda)) \\ \qquad - \operatorname{div} \hat{\mathcal{W}}^\varepsilon(x,\lambda) = \hat{\mathcal{F}}^\varepsilon(x,\lambda) \quad . \end{cases}$$

For fixed λ, $(5.8)_\varepsilon$ is just the homogenization problem for an elliptic operator with complex coefficients.

Upon inspection of (5.5), one sees that for every $t > 0$, $\mathcal{F}^\varepsilon$ converges to $\mathcal{F}$ in $C([0,t]; H^{-1}(\Omega))$ but that $\mathcal{F}^\varepsilon$ grows in H^{-1} at the same rate as u^ε does. Consequently, for each λ with $\operatorname{Re}\lambda > \lambda_0$, $\hat{\mathcal{F}}^\varepsilon(\lambda)$ converges to $\hat{\mathcal{F}}(\lambda)$ in $H^{-1}(\Omega; \mathbb{C})$. Similarly $\hat{u}_\varepsilon(\lambda)$ converges to $\hat{u}(\lambda)$ weakly in $H_0^1(\Omega; \mathbb{C})$. The sesquilinear form

(5.9) $$a_\varepsilon(\lambda;\ u,v) = \int_\Omega \hat{c}_\varepsilon(x,\lambda)A^\varepsilon(x)\nabla u(x)\overline{\nabla v}(x)dx$$

is continuous coercive on $H_0^1(\Omega;\ \mathbb{C})$ under the hypothesis

(5.10) $\mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda) \geq \rho_0(\lambda) > 0$, which we will check later.

Indeed, $\hat{c}_\varepsilon$ is bounded on Ω for each λ such that $\mathrm{Re}\ \lambda > 0$ so a_ε is continuous and for such λ's,

$$\mathrm{Re}\ a_\varepsilon(\lambda;\ u,u) = \int_\Omega \mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda)A^\varepsilon(x)|\nabla u(x)|^2dx \geq$$

$$\geq \alpha \int_\Omega \mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda)|\nabla u|^2dx$$

since A^ε is symmetric real coercive. Incidentally, another proof of existence for the solution u^ε is thus obtained by applying Lax-Milgram's theorem for $\hat{u}^\varepsilon$.

One can now apply a compactness result for complex homogenization (see for example Sanchez-Palencia [1], Murat [1] or Bensoussan-Lions-Papanicolaou [1]). For each λ with $\mathrm{Re}\ \lambda > \lambda_0$, there is a matrix-valued function $D(x,\lambda)$ (independent of $\hat{F}$ and $\hat{W}$) such that $(5.8)_\varepsilon$ implies at the limit $\varepsilon \to 0$:

(5.11) $$\begin{cases} -\ \mathrm{div}\ \hat{W} = \hat{F} \\ \hat{W}(x,\lambda) = D(x,\lambda)\nabla\hat{u}(x,\lambda) \end{cases} .$$

In order to apply the inverse Laplace transform to (5.11), all that is needed is that $D(x,\lambda)$ be analytic in λ with at most polynomial growth at $|\lambda| \to \infty$, in which case it is the Laplace transform of a distribution of finite order in t, denoted $E(x,t)$. That D is analytic is a mere consequence of the fact that it is a limit of a sequence of analytic functions of λ, the limit being locally uniform. From the uniform boundedness of c^ε and A^ε, one can conclude that $D(x,\lambda)$ is bounded by a multiple of $(\mathrm{Re}\ \lambda)^{-1}$. Consequently, $E(x,t)$ is a bounded distribution of order not more than 2, on $[0,+\infty[$, with values in the cone of bounded measurable symmetric

square matrices on Ω.

We now check that (5.10) holds: integration by parts in

$$\text{(5.12)} \qquad \mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda) = \int_0^\infty e^{-(\mathrm{Re}\ \lambda)t}\cos(\mathrm{Im}\ \lambda\ t)c_\varepsilon(t)dt$$

gives

$$\mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda) = \int_0^\infty(1 - \cos(t\ \mathrm{Im}\ \lambda))e^{-t\ \mathrm{Re}\ \lambda}(c_\varepsilon'' - 2\ \mathrm{Re}\ \lambda c_\varepsilon' + (\mathrm{Re}\ \lambda)^2 c_\varepsilon)dt\ .$$

Since $(1 - \cos(t\ \mathrm{Im}\ \lambda))$ and $-c_\varepsilon'$ are nonnegative functions and c_ε'' is a nonnegative measure,

$$\text{(5.13)} \qquad \mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda) > \int_0^{+\infty} (1 - \cos(t\ \mathrm{Im}\ \lambda))e^{-t\ \mathrm{Re}\ \lambda}(\mathrm{Re}\ \lambda)^2 c_\varepsilon(t)dt \quad ;$$

combining (5.12) and (5.13) one gets

$$\text{(5.14)} \qquad \mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda) > \frac{(\mathrm{Re}\ \lambda)^2}{1+(\mathrm{Re}\ \lambda)^2}\int_0^\infty e^{-t\ \mathrm{Re}\ \lambda} c_\varepsilon(t)dt \quad ;$$

But $c_\varepsilon(t) > \alpha$ implies with (5.14) that $\mathrm{Re}\ \hat{c}_\varepsilon(x,\lambda) > \alpha \dfrac{\mathrm{Re}\ \lambda}{1+(\mathrm{Re}\ \lambda)^2}$ which implies (5.10). Finally, we have proved the following theorem:

(5.15) <u>Theorem</u>: Let b_0^ε, c^ε, β^ε and $\sigma^\varepsilon = A^\varepsilon$ (linear case) satisfy hypotheses (4.4) with α and k independent of ε. There exists a sequence ε_n converging to zero, functions $b_0(x)$, $\beta(x,t)$ and a distribution $E(x,t)$ such that the solution u^{ε_n} of the corresponding problem (4.1), (4.2) converges in $C([0,T]; L^2(\Omega))$ to the solution u of

$$\text{(5.16)} \qquad b_0 u' - \mathrm{div}((E * \nabla u)') = G(u) \quad .$$

b_0 and β are the weak limits of $b_0^{\varepsilon_n}$ and β^{ε_n}, and E is obtained via its Laplace transform $D(x,\lambda)$ which is the complex elliptic homogenization of $\hat{c}_{\varepsilon_n}(x,\lambda)A^{\varepsilon_n}(x)$. ■

<u>Remark</u>: Even in the case of periodic problems, where there are explicit formulas for D it is not known whether $t \mapsto E(t,x)$ is in some appropriate sense, a convex decreasing function of t, not even whether it is a function of t, as one would suspect. This is one of the problem left open in the

theory, the other one being the homogenization of the non-linear case (where the Laplace-transform cannot be used).

Acknowledgment:

The authors are very grateful to Professor John Nohel who not only introduced them to non-linear Volterra equations in general and this problem in particular, but without whose help this paper would never have been written.

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1. REPORT NUMBER #2566	2. GOVT ACCESSION NO. A134427	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Homogenization for a Volterra Equation		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Hedy Attouch and Alain Damlamian		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE September 1983
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Volterra equation, homogenization, heat flow, heterogeneous material with memory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A model is given for the non-linear heat equation in a heterogeneous medium with memory. Its homogenization is carried out in two particular cases (including the linear one).

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